

Convex Quadric Surfaces

Valeriu Soltan

Department of Mathematical Sciences, George Mason University
Fairfax, Virginia 22030, USA
e-mail: vsoltan@gmu.edu

Abstract. We describe convex quadric surfaces in \mathbb{R}^n and characterize them as convex surfaces with quadric sections by a continuous family of hyperplanes.

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1 Introduction and main results

Characterizations of ellipses and ellipsoids among convex bodies in the plane or in space became an established topic of convex geometry on the turn of twenty's century. Comprehensive surveys on various characteristic properties of ellipsoids in the Euclidean space \mathbb{R}^n are given in [7] and [11] (see also [8]). Similar characterizations of unbounded convex quadrics, like paraboloids, sheets of elliptic hyperboloids or elliptic cones, are given by a short list of sporadic results (see, e. g., [1, 2, 13, 14]). Furthermore, even a classification of convex quadrics in \mathbb{R}^n is not established (although it is used in [13, 14] without proof). Our goal here is to describe convex quadrics in \mathbb{R}^n and to provide a characteristic property of these surfaces in terms of hyperplane sections.

We recall that a *quadric surface* (or a *second degree surface*) in \mathbb{R}^n , $n \geq 2$, is the locus of points $x = (\xi_1, \dots, \xi_n)$ that satisfy a quadratic equation

$$\sum_{i,k=1}^n a_{ik} \xi_i \xi_k + 2 \sum_{i=1}^n b_i \xi_i + c = 0, \quad (1.1)$$

where not all a_{ik} are zero. A *convex surface* is the boundary of an n -dimensional convex set distinct from \mathbb{R}^n . In particular, a hyperplane and a pair of parallel hyperplanes are convex surfaces. We say that a convex surface $S \subset \mathbb{R}^n$ is a *convex quadric* provided there is a real quadric surface $Q \subset \mathbb{R}^n$ and a convex component U of $\mathbb{R}^n \setminus Q$ such that S is the boundary of U . The following theorem plays a key role in the description of convex quadrics.

Theorem 1. *The complement of a real quadric surface $Q \subset \mathbb{R}^n$, $n \geq 2$, is the union of four or fewer open sets; at least one of these sets is convex if and only*

if the canonical form of Q is given by one of the equations

$$\begin{aligned} a_1\xi_1^2 + \cdots + a_k\xi_k^2 &= 1, \quad 1 \leq k \leq n, \\ a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 &= 1, \quad 2 \leq k \leq n, \\ a_1\xi_1^2 &= 0, \\ a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 &= 0, \quad 2 \leq k \leq n, \\ a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 &= \xi_k, \quad 2 \leq k \leq n, \end{aligned}$$

where all scalars a_i involved are positive.

Corollary 1. *A convex surface $S \subset \mathbb{R}^n$, $n \geq 2$, is a convex quadric if and only if there are Cartesian coordinates ξ_1, \dots, ξ_n such that S can be expressed by one of the equations*

$$\begin{aligned} a_1\xi_1^2 + \cdots + a_k\xi_k^2 &= 1, \quad 1 \leq k \leq n, \\ a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 &= 1, \quad \xi_1 \geq 0, \quad 2 \leq k \leq n, \\ a_1\xi_1^2 &= 0, \\ a_1\xi_1^2 - a_2\xi_2^2 - \cdots - a_k\xi_k^2 &= 0, \quad \xi_1 \geq 0, \quad 2 \leq k \leq n, \\ a_1\xi_1^2 + \cdots + a_{k-1}\xi_{k-1}^2 &= \xi_k, \quad 2 \leq k \leq n, \end{aligned}$$

where all scalars a_i involved are positive.

In what follows, a *plane* of dimension m is a translation of an m -dimensional subspace. We say that a plane L *properly* intersects an n -dimensional convex set K provided L intersects both the boundary $\text{bd } K$ and the interior $\text{int } K$ of K .

A well-known result of convex geometry states that the boundary of a convex body $K \subset \mathbb{R}^n$ is an ellipsoid if and only if there is a point $p \in \text{int } K$ such that all sections of $\text{bd } K$ by 2-dimensional planes through p are ellipses (see [3, 10] for $n = 3$ and [6, pp. 91–92] for $n \geq 3$). This result is generalized in [13] by showing that the boundary of an n -dimensional closed convex set $K \subset \mathbb{R}^n$ is a convex quadric if and only if there is a point $p \in \text{int } K$ such that all sections of $\text{bd } K$ by 2-dimensional planes through p are convex quadric curves. In this regard, we pose the following problem (solved in [5, 9] for the case of convex bodies): *Given an n -dimensional closed convex set $K \subset \mathbb{R}^n$ distinct from \mathbb{R}^n , $n \geq 3$, and a point $p \in \mathbb{R}^n$, is it true that either $\text{bd } K$ is a convex quadric or K is a convex cone with apex p provided all proper sections of $\text{bd } K$ by 2-dimensional planes through p are convex quadric curves?*

Kubota [10] proved that, given a pair of bounded convex surfaces in \mathbb{R}^3 , one being enclosed by the other, if all planar sections of the biggest surface by planes tangent to the second surface are ellipses, then the biggest surface is an ellipsoid. Independently, Bianchi and Gruber [4] established the following far-reaching assertion: If K is a convex body in \mathbb{R}^n , $n \geq 3$, and $\delta(u)$ is a continuous real-valued function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ such that for each vector $u \in S^{n-1}$ the hyperplane $H(u) = \{x \mid x \cdot u = \delta(u)\}$ intersects $\text{bd } K$ along an

$(n - 1)$ -dimensional ellipsoid, then $\text{bd } K$ is an ellipsoid. Our second theorem extends this assertion to the case of n -dimensional closed convex sets.

Theorem 2. *Let $K \subset \mathbb{R}^n$ be an n -dimensional closed convex set distinct from \mathbb{R}^n , $n \geq 3$, and $\delta(u)$ be a continuous real-valued function on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ such that for each vector $u \in S^{n-1}$ the hyperplane $H(u) = \{x \mid x \cdot u = \delta(u)\}$ either lies in K or intersects $\text{bd } K$ along an $(n - 1)$ -dimensional convex quadric. Then $\text{bd } K$ is a convex quadric.*

In what follows, the origin of \mathbb{R}^n is denoted o . We say that a plane L supports a closed convex set K provided L intersects K such that $L \cap \text{int } K = \emptyset$. The recession cone of K is defined by

$$\text{rec } K = \{y \in \mathbb{R}^n \mid x + \alpha y \in K \text{ for all } x \in K \text{ and } \alpha \geq 0\}.$$

It is well-known that $\text{rec } K \neq \{o\}$ if and only if K is unbounded; K is called *line-free* if it contains no line. Finally, $\text{rint } M$ and $\text{rbd } M$ denote the relative interior and the relative boundary of a convex set $M \subset \mathbb{R}^n$.

2 Auxiliary Lemmas

The proof of Theorem 1 uses a description of certain quadric surfaces in \mathbb{R}^n as consecutive revolutions of lower-dimensional quadrics. To describe these revolutions, choose any subspaces L_1, L_2 , and L_3 of \mathbb{R}^n such that $L_1 \subset L_2 \subset L_3$ and

$$\dim L_1 = m - 1, \quad \dim L_2 = m, \quad \dim L_3 = m + 1, \quad 2 \leq m \leq n - 1.$$

Let M be the 2-dimensional subspace of L_3 orthogonal to L_1 . Given a point $y \in L_2$, put $M_y = y + M$ and denote by z the point of intersection of L_1 and M_y (obviously, z is the orthogonal projection of y on L_1). Let C_y be the circumference in M_y with center z and radius $\|y - z\|$. We say that a set $X \subset L_3$ is the *revolution* of a set $Y \subset L_2$ about L_1 within L_3 provided $X = \cup\{C_y \mid y \in Y\}$. A set $Z \subset \mathbb{R}^n$ is called *symmetric* about a subspace $N \subset \mathbb{R}^n$ provided for any point $x \in Z$ and its orthogonal projection u on N , the point $2u - x$ lies in Z . In these terms, we formulate three lemmas, the first one being obvious.

Lemma 1. *If a set $Y \subset L_2$ is symmetric about L_1 and X is the revolution of Y about L_1 within L_3 , then X is symmetric about L_2 and any component of X is the revolution of a suitable component of Y about L_1 within L_3 . \square*

In what follows, $\langle e_1, \dots, e_k \rangle$ means the span of vectors $e_1, \dots, e_k \in \mathbb{R}^n$.

Lemma 2. *If a set $Y \subset L_2$ is symmetric about L_1 and X is the revolution of Y about L_1 within L_3 , then X is a convex set if and only if Y is a convex set.*

Proof. Without loss of generality, we may put $L_3 = \mathbb{R}^n$. Choose an orthonormal basis e_1, \dots, e_n for \mathbb{R}^n such that

$$L_1 = \langle e_1, \dots, e_{n-2} \rangle \quad \text{and} \quad L_2 = \langle e_1, \dots, e_{n-1} \rangle.$$

Clearly, $x = (\xi_1, \dots, \xi_n)$ belongs to X if and only if there is a point

$$y = (\xi_1, \dots, \xi_{n-2}, \xi'_{n-1}, 0) \in Y \quad \text{where} \quad \xi'_{n-1} = \sqrt{\xi_{n-1}^2 + \xi_n^2}.$$

If X is convex, then Y is convex due to $Y = X \cap L_2$. Let Y be convex. Choose any points $a = (\alpha_1, \dots, \alpha_n)$ and $b = (\beta_1, \dots, \beta_n)$ in X and a scalar $\lambda \in [0, 1]$. We intend to show that $c = (1 - \lambda)a + \lambda b \in X$. Let

$$a' = (\alpha_1, \dots, \alpha_{n-2}, \alpha'_{n-1}, 0), \quad b' = (\beta_1, \dots, \beta_{n-2}, \beta'_{n-1}, 0),$$

and

$$c' = ((1 - \lambda)\alpha_1 + \lambda\beta_1, \dots, (1 - \lambda)\alpha_{n-2} + \lambda\beta_{n-2}, (1 - \lambda)\alpha'_{n-1} + \lambda\beta'_{n-1}, 0)$$

be points in Y where

$$\alpha'_{n-1} = \sqrt{\alpha_{n-1}^2 + \alpha_n^2}, \quad \text{and} \quad \beta'_{n-1} = \sqrt{\beta_{n-1}^2 + \beta_n^2}.$$

Then $a', b' \in Y$, and $c' = (1 - \lambda)a' + \lambda b' \in Y$ due to convexity of Y . Because Y is symmetric about L_1 , we have

$$((1 - \lambda)\alpha_1 + \lambda\beta_1, \dots, (1 - \lambda)\alpha_{n-2} + \lambda\beta_{n-2}, \mu, 0) \in Y$$

for any scalar μ with $|\mu| \leq (1 - \lambda)\alpha'_{n-1} + \lambda\beta'_{n-1}$. Let

$$y = ((1 - \lambda)\alpha_1 + \lambda\beta_1, \dots, (1 - \lambda)\alpha_{n-2} + \lambda\beta_{n-2}, \rho, 0),$$

where

$$\rho = \sqrt{((1 - \lambda)\alpha_{n-1} + \lambda\beta_{n-1})^2 + ((1 - \lambda)\alpha_n + \lambda\beta_n)^2}.$$

From $\alpha_{n-1}\beta_{n-1} + \alpha_n\beta_n \leq \alpha'_{n-1}\beta'_{n-1}$, we obtain $\rho \leq (1 - \lambda)\alpha'_{n-1} + \lambda\beta'_{n-1}$, which gives $y \in Y$. Clearly, the point

$$z = ((1 - \lambda)\alpha_1 + \lambda\beta_1, \dots, (1 - \lambda)\alpha_{n-2} + \lambda\beta_{n-2}, 0, 0)$$

is the orthogonal projection of y on L_1 . The equalities $\|c - z\| = \|y - z\| = \rho$ imply that $c \in C_y \subset X$. Hence X is convex. \square

Let $Q \subset \mathbb{R}^n$ be a real quadric surface. A suitable choice of Cartesian coordinates transforms (1.1) into one of the following canonical forms

$$\begin{aligned} A_k &: \xi_1^2 + \dots + \xi_k^2 = 1, \quad 1 \leq k \leq n, \\ B_{k,r} &: \xi_1^2 + \dots + \xi_k^2 - \xi_{k+1}^2 - \dots - \xi_r^2 = 1, \quad 1 \leq k < r \leq n, \\ C_k &: \xi_1^2 + \dots + \xi_k^2 = 0, \quad 1 \leq k \leq n, \\ D_{k,r} &: \xi_1^2 + \dots + \xi_k^2 - \xi_{k+1}^2 - \dots - \xi_r^2 = 0, \quad 1 \leq k < r \leq n, \\ E_{k,r} &: \xi_1^2 + \dots + \xi_k^2 - \xi_{k+1}^2 - \dots - \xi_{r-1}^2 = \xi_r, \quad 1 \leq k < r \leq n. \end{aligned}$$

Lemma 3. *Within \mathbb{R}^n , $n \geq 3$,*

- 1) A_n is the revolution of $A_{n-1} \subset \langle e_1, \dots, e_{n-1} \rangle$ about $\langle e_1, \dots, e_{n-2} \rangle$,
- 2) $B_{k,n}$ is the revolution of $B_{k,n-1} \subset \langle e_1, \dots, e_{n-1} \rangle$ about $\langle e_1, \dots, e_{n-2} \rangle$,
 $1 \leq k \leq n-2$,
- 3) $D_{k,n}$ is the revolution of $D_{k,n-1} \subset \langle e_1, \dots, e_{n-1} \rangle$ about $\langle e_1, \dots, e_{n-2} \rangle$,
 $1 \leq k \leq n-2$,
- 4) $B_{k,n}$ is the revolution of $B_{k-1,n-1} \subset \langle e_2, \dots, e_n \rangle$ about $\langle e_3, \dots, e_n \rangle$, $2 \leq k \leq n-1$,
- 5) $D_{k,n}$ is the revolution of $D_{k-1,n-1} \subset \langle e_2, \dots, e_n \rangle$ about $\langle e_3, \dots, e_n \rangle$, $2 \leq k \leq n-1$.

Proof. 1) Given a point $x = (\xi_1, \dots, \xi_n) \in A_n$, put

$$y = (\xi_1, \dots, \xi_{n-2}, \sqrt{\xi_{n-1}^2 + \xi_n^2}, 0), \quad z = (\xi_1, \dots, \xi_{n-2}, 0, 0). \quad (2.1)$$

Then $y \in A_{n-1} \subset \langle e_1, \dots, e_{n-1} \rangle$ and z is the orthogonal projection of y on $\langle e_1, \dots, e_{n-2} \rangle$. From

$$\|x - z\| = \|y - z\| = \sqrt{\xi_{n-1}^2 + \xi_n^2}$$

we see that $x \in C_y$. So, A_n lies in the revolution of A_{n-1} about $\langle e_1, \dots, e_{n-2} \rangle$. Conversely, if $y = (\eta_1, \dots, \eta_{n-1}, 0)$ is a point in $A_{n-1} \subset \langle e_1, \dots, e_{n-1} \rangle$ and $z = (\eta_1, \dots, \eta_{n-2}, 0, 0)$ is the orthogonal projection of y on $\langle e_1, \dots, e_{n-2} \rangle$, then any point u from the circle $C_y \subset y + \langle e_{n-1}, e_n \rangle$ can be written as

$$u = (\eta_1, \dots, \eta_{n-2}, \gamma_{n-1}, \gamma_n), \quad \text{where} \quad \gamma_{n-1}^2 + \gamma_n^2 = \eta_{n-1}^2.$$

Clearly, $u \in A_n$, which shows that A_n contains the revolution of A_{n-1} about $\langle e_1, \dots, e_{n-2} \rangle$.

Cases 2)–5) are considered similarly, where the points y and z are defined, respectively, by (2.1) in cases 2) and 3), and by

$$y = (0, \sqrt{\xi_1^2 + \xi_2^2}, \xi_3, \dots, \xi_n), \quad z = (0, 0, \xi_3, \dots, \xi_n)$$

in cases 4) and 5). □

3 Proof of Theorem 1

Let $Q \subset \mathbb{R}^n$ be a real quadric surface. We may suppose that Q has one of the forms A_k , $B_{k,r}$, C_k , $D_{k,r}$, $E_{k,r}$ described above. First, we exclude the trivial cases $Q = A_1$ (when Q is a pair of parallel hyperplanes) and $Q = C_k$ (when Q is an $(n-k)$ -dimensional subspace). Furthermore, we can reduce the proof to

the case when Q has one of the forms $A_n, B_{k,n}, D_{k,n}, E_{k,n}$. Indeed, if $k < n$ or $r < n$, then Q is a both-way unbounded cylinder, that is, it is the Cartesian product of a subspace \mathbb{R}^k (respectively, \mathbb{R}^r) and a quadric P of the same type in the orthogonal complement \mathbb{R}^{n-k} (respectively, \mathbb{R}^{n-r}); clearly, Q satisfies the conclusion of the theorem if and only if P does.

Our further consideration is organized by induction on n . The cases $n = 2$ and $n = 3$ follow immediately from the well-known properties of quadric curves and surfaces. Suppose that $n \geq 4$. Assuming that the conclusion of Theorem 1 holds for all $m < n$, let the quadric surface $Q \subset \mathbb{R}^n$ have one of the forms $A_n, B_{k,n}, D_{k,n}, E_{k,n}$. We consider these forms separately.

Case 1. Let $Q = A_n$. By Lemma 3, A_n can be obtained from

$$A_2 = \{(\xi_1, \xi_2) \mid \xi_1^2 + \xi_2^2 = 1\} \subset \langle e_1, e_2 \rangle$$

by consecutive revolutions of $A_i \subset \langle e_1, \dots, e_i \rangle$ about $\langle e_1, \dots, e_{i-1} \rangle$ within the subspace $\langle e_1, \dots, e_{i+1} \rangle$, $i = 2, \dots, n-1$. Since both components of $\langle e_1, e_2 \rangle \setminus A_2$ are symmetric about the line $\langle e_1 \rangle$, Lemmas 1 and 2 imply that $\mathbb{R}^n \setminus A_n$ consists of two components; one of them, given by $\xi_1^2 + \dots + \xi_n^2 < 1$, is convex.

Case 2. Let $Q = B_{k,n}$, $1 \leq k \leq n-1$. If $k = 1$, then Lemma 3 implies that $B_{1,n}$ can be obtained from

$$B_{1,2} = \{(\xi_1, \xi_2) \mid \xi_1^2 - \xi_2^2 = 1\} \subset \langle e_1, e_2 \rangle$$

by consecutive revolutions of $B_{1,i} \subset \langle e_1, \dots, e_i \rangle$ about $\langle e_1, \dots, e_{i-1} \rangle$ within the subspace $\langle e_1, \dots, e_{i+1} \rangle$, $i = 2, \dots, n-1$. Since all three components of $\langle e_1, e_2 \rangle \setminus B_{1,2}$ are symmetric about the line $\langle e_1 \rangle$, Lemmas 1 and 2 imply that $\mathbb{R}^n \setminus B_{1,n}$ consists of three components; two of them, given, respectively, by

$$\xi_1 > \sqrt{\xi_2^2 + \dots + \xi_n^2 + 1} \quad \text{and} \quad \xi_1 < -\sqrt{\xi_2^2 + \dots + \xi_n^2 + 1},$$

are convex. If $k \geq 2$, then $B_{k,n}$ can be obtained from

$$B_{1,2} = \{(\xi_k, \xi_{k+1}) \mid \xi_k^2 - \xi_{k+1}^2 = 1\} \subset \langle e_k, e_{k+1} \rangle$$

in two steps. First, we obtain $B_{k,k+1} \subset \mathbb{R}^{k+1} = \langle e_1, \dots, e_{k+1} \rangle$ by consecutive revolutions of $B_{i,i+1} \subset \langle e_{k+1-i}, e_{k+2-i}, \dots, e_{k+1} \rangle$ about $\langle e_{k+2-i}, \dots, e_{k+1} \rangle$ within $\langle e_{k-i}, e_{k+1-i}, \dots, e_{k+1} \rangle$, $i = 1, 2, \dots, k-1$. The complement of

$$B_{2,3} = \{(\xi_{k-1}, \xi_k, \xi_{k+1}) \mid \xi_{k-1}^2 + \xi_k^2 - \xi_{k+1}^2 = 1\}$$

in $\langle e_{k-1}, e_k, e_{k+1} \rangle$, consists of two components, both symmetric about $\langle e_k, e_{k+1} \rangle$. Since none of these components is convex, Lemmas 1 and 2 imply that $\mathbb{R}^{k+1} \setminus B_{k,k+1}$ consists of two components, both symmetric about any k -dimensional coordinate subspace of \mathbb{R}^{k+1} , but none of them convex.

Second, we obtain $B_{k,n}$ from $B_{k,k+1}$ by consecutive revolutions of $B_{k,j} \subset \langle e_1, \dots, e_j \rangle$ about $\langle e_1, \dots, e_{j-1} \rangle$ within $\langle e_1, \dots, e_{j+1} \rangle$, $j = k+1, \dots, n-1$. As above, $\mathbb{R}^n \setminus B_{k,n}$ consists of two components, none of them convex.

Case 3. Let $Q = D_{k,n}$, $1 \leq k \leq n-1$. If $k = 1$, then $D_{1,n}$ can be obtained from

$$D_{1,2} = \{(\xi_1, \xi_2) \mid \xi_1^2 - \xi_2^2 = 0\} \subset \langle e_1, e_2 \rangle$$

by consecutive revolutions of $D_{1,i} \subset \langle e_1, \dots, e_i \rangle$ about $\langle e_1, \dots, e_{i-1} \rangle$ within the subspace $\langle e_1, \dots, e_{i+1} \rangle$, $i = 2, \dots, n-1$. The complement of

$$D_{1,3} = \{(\xi_1, \xi_2, \xi_3) \mid \xi_1^2 - \xi_2^2 + \xi_3^2 = 0\}$$

in $\langle e_1, e_2, e_3 \rangle$ consists of tree components, all symmetric about $\langle e_1, e_2 \rangle$. Since two of these components are convex, Lemmas 1 and 2 imply that $\mathbb{R}^n \setminus D_{1,n}$ consists of three components; two of them, given, respectively, by

$$\xi_1 > \sqrt{\xi_2^2 + \dots + \xi_n^2} \quad \text{and} \quad \xi_1 < -\sqrt{\xi_2^2 + \dots + \xi_n^2},$$

are convex.

Since the case $k = n-1$ is reducible to that of $k = 1$ (by reordering e_1, e_2, \dots, e_n as e_n, e_{n-1}, \dots, e_1), it remains to assume that $2 \leq k \leq n-2$. Then $D_{k,n}$ can be obtained from

$$D_{2,3} = \{(\xi_{k-1}, \xi_k, \xi_{k+1}) \mid \xi_{k-1}^2 + \xi_k^2 - \xi_{k+1}^2 = 0\} \subset \langle e_{k-1}, e_k, e_{k+1} \rangle$$

in two steps. First, we obtain $D_{2,n-k+2} \subset \langle e_{k-1}, e_k, \dots, e_n \rangle$ by consecutive revolutions of $D_{2,i} \subset \langle e_{k-1}, e_k, \dots, e_i \rangle$ about $\langle e_{k-1}, e_k, \dots, e_{i-1} \rangle$ within $\langle e_{k-1}, e_k, \dots, e_{i+1} \rangle$, $i = k+1, \dots, n-1$. Clearly, $\langle e_{k-1}, e_k, e_{k+1} \rangle \setminus D_{2,3}$ consists of three components; two of them,

$$\xi_{k+1} > \sqrt{\xi_{k-1}^2 + \xi_k^2} \quad \text{and} \quad -\xi_{k+1} < \sqrt{\xi_{k-1}^2 + \xi_k^2},$$

are convex and symmetric to each other about $\langle e_{k-1}, e_k \rangle$. Hence $\langle e_{k-1}, e_k, e_{k+1}, e_{k+2} \rangle \setminus D_{3,4}$ consists of two components, none of them convex. Lemmas 1 and 2 imply that $\mathbb{R}^{n-k+2} \setminus D_{2,n-k+2}$ consists of two components, none of them convex.

Next, we obtain $D_{k,n}$ from $D_{2,n-k+2}$ by consecutive revolutions of the surface $D_{i,n-k+i} \subset \langle e_{k-i+1}, \dots, e_n \rangle$ about $\langle e_{k-i+2}, \dots, e_n \rangle$ within $\langle e_{k-i}, \dots, e_n \rangle$, $i = 2, \dots, k-1$. As above, $\mathbb{R}^n \setminus D_{k,n}$ consists of two components, none of them convex.

Case 4. Let $Q = E_{k,n}$, $1 \leq k \leq n-1$. Clearly, $E_{k,n}$ is the graph of a real-valued function φ on $\mathbb{R}^{n-1} = \langle e_1, \dots, e_{n-1} \rangle$, given by

$$\xi_n = \varphi(\xi_1, \dots, \xi_{n-1}) = \xi_1^2 + \dots + \xi_k^2 - \xi_{k+1}^2 - \dots - \xi_{n-1}^2.$$

Hence $\mathbb{R}^n \setminus E_{k,n}$ has two components. The Hessian $\left(\frac{\partial^2 \varphi}{\partial \xi_i \partial \xi_j} \right)$ is a diagonal $n \times n$ -matrix, with 2's on its first k diagonal positions and -2 's on the other $n-k-1$ diagonal positions. Therefore, φ is not concave, being convex if and only if $k = n-1$. So, $\mathbb{R}^n \setminus E_{k,n}$ has a convex component if and only if $k = n-1$; it is given by $\xi_1^2 + \dots + \xi_{n-1}^2 < \xi_n$.

4 Proof of Theorem 2

Under the assumption of Theorem 2, we divide the proof into a sequence of lemmas.

Lemma 4. *If K contains a line, then $\text{bd } K$ is a both-way unbounded convex quadric cylinder.*

Proof. If l is a line in K , then K is the direct sum $\langle u_0 \rangle \oplus (K \cap H(u_0))$, where $\langle u_0 \rangle$ is the 1-dimensional subspace spanned by a unit vector u_0 parallel to l . By the assumption, $\text{bd } K \cap H(u_0)$ is an $(n-1)$ -dimensional convex quadric. Hence $\text{bd } K = \langle u_0 \rangle \oplus (\text{bd } K \cap H(u_0))$ is a both-way unbounded convex quadric cylinder. \square

Due to Lemma 4, we may further assume that K is line-free. Then no hyperplane lies in K ; so, every hyperplane $H(u)$, $u \in S^{n-1}$, properly intersects K .

Lemma 5. *For any $(n-2)$ -dimensional plane L supporting K , there is a hyperplane $H(u)$, $u \in S^{n-1}$, that contains L .*

Proof. Let P be the 2-dimensional subspace orthogonal to L and π be the orthogonal projection of \mathbb{R}^n onto P . The intersection $L \cap P$ is a singleton, say $\{v\}$. The set $M = \pi(K)$ is convex, $\text{rint } M = \pi(\text{int } K)$, and $v \in \text{rbd } M$. Denote by l a line in P that supports M at v , and let u_1, u_2 be unit vectors in P such that u_1 is parallel to l and u_2 is orthogonal to l , where $v + u_2$ is an outward unit normal to M at v . Without loss of generality, we may suppose that u_2 lies between u_1 and $-u_1$ according to the clockwise bypass of $P \cap S^{n-1}$.

Assume, for contradiction, that no line $l(u) = P \cap H(u)$, $u \in P \cap S^{n-1}$, contains v . We consider the cases $v = o$ and $v \neq o$ separately.

If $v = o$, then $\delta(u) \neq 0$ for all $u \in P \cap S^{n-1}$. If $\delta(u_1) > 0$, then the continuous curve $\delta(u)u$ with endpoints $\delta(u_1)u_1$ and $\delta(-u_1)(-u_1)$, obtained by the clockwise bypass of $P \cap S^{n-1}$, entirely lies in the closed halfplane of P bounded by l and disjoint from $\text{rint } M$. In particular, the line $l(u_2)$ is parallel to l and disjoint from $\text{rint } M$, contradicting the assumption $\text{int } K \cap H(u_2) \neq \emptyset$. If $\delta(u_1) < 0$, we similarly obtain a contradiction with $\text{int } K \cap H(-u_2) \neq \emptyset$ by considering the counterclockwise bypass of $P \cap S^{n-1}$ from u_1 to $-u_1$.

Let $v \neq o$. Denote by C the circle in P with diameter $[o, v]$. Clearly, $\delta(u)u \notin C$ for all $u \in P \cap S^{n-1}$ due to the assumption that no line $l(u)$ contains v . Considering separately the cases $C \cap l = \{v\}$ and $C \cap l \neq \{v\}$, we obtain, similarly to the case $v = o$ above, that $\text{rint } M \cap l(u_2) = \emptyset$ or $\text{rint } M \cap l(-u_2) = \emptyset$, which is impossible. \square

We recall that an n -dimensional closed convex set $K \subset \mathbb{R}^n$ distinct from \mathbb{R}^n is *strictly convex* if $\text{bd } K$ contains no line segments; K is called *regular* provided any point $x \in \text{bd } K$ belongs to a unique hyperplane supporting K .

Lemma 6. *If K is neither strictly convex nor regular, then $\text{bd } K$ is a sheet of elliptic cone.*

Proof. First, we are going to show that if K is not regular, then K is not strictly convex. Indeed, suppose that K is not regular and choose a singular point $x \in \text{bd } K$. Let G_1 and G_2 be distinct hyperplanes both supporting K at x , and G be a hyperplane through $G_1 \cap G_2$ supporting K and distinct from both G_1 and G_2 . Choose in G an $(n-2)$ -dimensional plane L through x which is distinct from $G_1 \cap G_2$. By Lemma 5, there is a hyperplane $H(u)$, $u \in S^{n-1}$, containing L . Because $H(u)$ meets $\text{int } K$, the point x is singular for the $(n-1)$ -dimensional convex surface $E(u) = \text{bd } K \cap H(u)$. Hence $E(u)$ must be a sheet of elliptic cone. Choosing a line segment in $E(u)$, we conclude that K is not strictly convex.

Now, assume that K is not strictly convex and choose a line segment $[x, z] \subset \text{bd } K$. By Lemma 5, there is a hyperplane $H(u_0)$, $u_0 \in S^{n-1}$, containing the line through x and z . Since the $(n-1)$ -dimensional convex quadric $E(u_0) = \text{bd } K \cap H(u_0)$ is line-free and not strictly convex, it should be a sheet of elliptic cone. Let v be the apex of $E(u_0)$. Denote by h_1 the halfline $[v, x)$ and choose another halfline $h_2 = [v, w) \subset E(u_0)$ such that the 2-dimensional plane through $h_1 \cup h_2$ intersects $\text{int } K$ (this is possible since $H(u_0)$ meets $\text{int } K$). Let G_2 be a hyperplane supporting K with the property $h_2 \subset G_2$. By the above, $h_1 \not\subset G_2$.

Choose a halfline h with apex v tangent to K and so close to h_1 that $h \not\subset G_2$. Let G be a hyperplane through h that supports K . By Lemma 5, there is a hyperplane $H(u)$, $u \in S^{n-1}$, that meets $\text{int } K$ and contains h . Since the section $E(u) = \text{bd } K \cap H(u)$ is bounded by both G and G_2 , the point v is singular for $E(u)$. As above, $E(u)$ is a sheet of elliptic cone. Hence $h \subset \text{bd } K$. Varying h and h_2 , we obtain by the argument above that every tangent halfline of K at v lies in $\text{bd } K$. This shows that K is a convex cone with apex v . Finally, choose a hyperplane $H(u_1)$, $u_1 \in S^{n-1}$, that properly intersects K along a bounded set (this is possible since K is line-free). By the assumption, $\text{bd } K \cap H(u_1)$ is an $(n-1)$ -dimensional ellipsoid. So, $\text{bd } K$ is a sheet elliptic cone with apex v generated by $\text{bd } K \cap H(u_1)$. \square

Lemma 7. *Let K be strictly convex and regular. There is a scalar $\rho > 0$ with the property that for any vector $u \in S^{n-1}$ and a point $x \in \text{bd } K \cap H(u)$ there is a ball $B_\rho(z)$ of radius ρ centered at a point $z \in H(u)$ such that $B_\rho(z) \cap H(u) \subset K \cap H(u)$ and $x \in \text{bd } B_\rho(z)$.*

Proof. Since $\delta(u)$ is continuous on S^{n-1} , the set $\Delta = \{\delta(u)u \mid u \in S^{n-1}\}$ is compact.

Case 1. Assume that K is bounded and denote by d the diameter of K . By a compactness argument, there is a scalar $\gamma > 0$ with the following property: if $G(u)$ is a hyperplane parallel to $H(u)$ and supporting K , then the distance between $G(u)$ and $H(u)$ is at least γ . Choose a point $x \in \text{int } K$ and a closed ball $B_\mu(x) \subset K$ of radius $\mu > 0$. Fix a vector $u \in S^{n-1}$ and consider the $(n-1)$ -dimensional ellipsoid $E(u) = \text{bd } K \cap H(u)$. Let $G(u)$ be one of the hyperplanes parallel to $H(u)$ and supporting K such that x lies either in $H(u)$ or between $H(u)$ and $G(u)$. Choose a point $v \in K \cap G(u)$ and denote by C the convex cone with apex v generated by $B_\mu(x)$. Clearly, $F(u) = C \cap H(u)$ is an $(n-1)$ -dimensional ellipsoid that lies in $E(u)$. Because $\|x - v\| \leq d$ and the distance

between $H(u)$ and $G(u)$ is at least γ , there is a scalar $\varepsilon > 0$, not depending on u , such that $F(u)$ contains an $(n-1)$ -dimensional ball $B_\varepsilon(w) \cap H(u)$, with $w \in H(u)$. Since the diameter of $E(u)$ is less than or equal to d , there is a scalar $\rho > 0$ that satisfies the condition of the lemma.

Case 2. Assume that K is unbounded. Because Δ is compact, we can choose a closed halfspace V whose boundary hyperplane G is so far from Δ that the following conditions are satisfied:

- a) $K_1 = K \cap V$ is a convex body and $\Delta \subset \text{int } V$,
- b) any $(n-1)$ -dimensional convex quadric $E(u) = \text{bd } K \cap H(u)$, $u \in S^{n-1}$, has a vertex in $\text{int } V$,
- c) any unbounded $(n-1)$ -dimensional convex quadric $E(u) = \text{bd } K \cap H(u)$, $u \in S^{n-1}$, intersects the relative interior of $K \cap G$.

Fix a vector $u \in S^{n-1}$ and consider the $(n-1)$ -dimensional convex quadric $E(u) = \text{bd } K \cap H(u)$. If $E(u)$ is an ellipsoid, then we proceed similarly to Case 1, by choosing a ball $B_{\varepsilon_1}(z)$ of radius $\varepsilon_1 > 0$ centered at a point $z \in H(u)$ such that $B_{\varepsilon_1}(z) \cap H(u) \subset K_1 \cap H(u)$. So, the lemma holds provided $E(u)$ is bounded.

Now suppose that $E(u)$ is unbounded. By condition b), the vertex of $E(u)$ lies in $\text{int } K_1$. Denote by $r(u)$ the radius of the largest ball $B_{r(u)}(z)$, $z \in H(u)$, such that $B_{r(u)}(z) \cap H(u) \subset K \cap H(u)$ and the vertex of $E(u)$ belongs to $\text{bd } B_{r(u)}(z)$. Clearly, any other point $x \in E(u)$ lies in the relative boundary of an $(n-1)$ -dimensional ball from $K \cap H(u)$ of radius $r(u)$. Assume, for contradiction, that the conclusion of Lemma 7 does not hold. Then we can find a sequence of unit vectors $u_1, u_2, \dots \in S^{n-1}$ such that the unbounded convex quadrics $E(u_1), E(u_2), \dots$ satisfy the condition $r(u_k) \rightarrow 0$ as $k \rightarrow \infty$. Since the vertices of $E(u_k)$ belong to K_1 , we conclude, by a compactness argument, the existence of a subsequence $E(u_{k_1}), E(u_{k_2}), \dots$, that converges to a convex quadric $E(u)$ with $r(u) = 0$, which is a sheet of elliptic cone. The last is impossible since K is strictly convex. \square

Lemma 8. *Let K be strictly convex and regular. There are hyperplanes $H(u_1)$ and $H(u_2)$, $u_1, u_2 \in S^{n-1}$, such that both sections $\text{bd } K \cap H(u_1)$ and $\text{bd } K \cap H(u_2)$ are $(n-1)$ -dimensional ellipsoids whose intersection is an $(n-2)$ -dimensional ellipsoid.*

Proof. Since K is line-free, there is a 2-dimensional subspace P such that the orthogonal projection, M , of K onto P is a line-free closed convex set (see, e.g., [12]). Denote by \mathcal{F} the family of lines $l(u) = P \cap H(u)$, $u \in P \cap S^{n-1}$, such that $l(u) \cap M$ is bounded. Let $l(u_0)$ be one of these lines. Put $[v, w] = l(u_0) \cap M$. Clearly, $l(u_0)$ cuts M into 2-dimensional closed convex subsets, M' and M'' , at least one of them, say M' , being compact. If there is a line $l(u) \in \mathcal{F}$ which intersects the open line segment $]v, w[$, then the hyperplanes $H(u)$ and $H(u_0)$ have the desired property. Assume that no line $l(u) \in \mathcal{F}$ intersects $]v, w[$. Then no line $l(u) \in \mathcal{F}$ intersects $\text{rint } M'$. Indeed, if a line $l(u_1) \in \mathcal{F}$ intersected

rint M' , then, continuously rotating $u \in P \cap S^{n-1}$ from the initial position u_1 , we would find a line $l(u_2)$ supporting M at v or at w (which is impossible since $\text{int } K \cap H(u_2) \neq \emptyset$). This argument shows that M'' is unbounded, since otherwise we repeat the consideration above for M'' .

Rotating $u \in P \cap S^{n-1}$ counterclockwise from the initial position u_0 , we observe that the lines $l(u) \in \mathcal{F}$ cover the whole unbounded branch of $\text{rbd } M''$ with endpoint v . Similarly, the lines $l(u) \in \mathcal{F}$ cover the second unbounded branch of $\text{rbd } M''$, with endpoint w . This implies the existence of lines $l(u_3), l(u_4) \in \mathcal{F}$ such that the line segments $l(u_3) \cap K$ and $l(u_4) \cap K$ have a common interior point. Clearly, the respective ellipsoids $\text{bd } K \cap H(u_3)$ and $\text{bd } K \cap H(u_4)$ satisfy the conclusion of the lemma. \square

Lemma 9. *Let K be strictly convex and regular. If $\text{bd } K$ contains an open piece of real quadric surface, then $\text{bd } K$ is a convex quadric.*

Proof. Let A be an open piece of real quadric surface $Q \subset \mathbb{R}^n$ which lies in $\text{bd } K$. We state that $\text{bd } K \subset Q$. Assume, for contradiction, that $\text{bd } K \not\subset Q$, and choose a maximal (under inclusion) open piece B of $\text{bd } K \cap Q$ that contains A . Let $U_r(x) \subset \mathbb{R}^n$ be an open ball with center $x \in B$ and radius $r > 0$ such that $\text{bd } K \cap U_r(x) \subset B$. Continuously moving x towards $\text{bd } K \setminus B$, we find points $x_0 \in B$ and $z \in \text{bd } K \setminus B$ with the property $\text{bd } K \cap U_r(x_0) \subset B$ and $\|x_0 - z\| = r$.

Let G be the hyperplane through z which supports K (G is unique since K is regular). Denote by \mathcal{L} the family of $(n-2)$ -dimensional planes $L \subset G$ that contain z and are distinct from the $(n-2)$ -dimensional plane $L_0 \subset G$ tangent to $U_r(x_0) \cap G$ at z . By Lemma 5, any plane $L \in \mathcal{L}$ lies in a respective hyperplane $H_L(u)$. Due to Lemma 7, there is a scalar $t > 0$ so small that the union of $(n-1)$ -dimensional convex quadrics $E_L(u) = \text{bd } K \cap H_L(u)$, $L \in \mathcal{L}$, is dense in the surface t -neighborhood $\text{bd } K \cap U_t(z)$ of z . Clearly, each $E_L(u)$ has a nontrivial strictly convex intersection with B . Since $E_L(u)$ is a unique convex quadric containing $E_L(u) \cap B$, we conclude that $E_L(u) \subset Q$. By continuity,

$$\text{bd } K \cap U_t(z) \subset \text{cl}(\cup\{E_L(u) \mid L \in \mathcal{L}\}) \subset Q.$$

Hence $\text{bd } K \cap U_t(z) \subset B$, contrary to the choice of $z \in \text{bd } K \setminus B$. Thus $\text{bd } K \subset Q$. Because $\text{int } K$ is a convex component of $\mathbb{R}^n \setminus Q$, the surface $\text{bd } K$ is a convex quadric. \square

Lemma 10. *Let E_1 and E_2 be $(n-1)$ -dimensional ellipsoids in \mathbb{R}^n , $n \geq 3$, which lie, respectively, in hyperplanes H_1 and H_2 of \mathbb{R}^n such that $E = E_1 \cap E_2$ is an $(n-2)$ -dimensional ellipsoid. For any point $v \in \mathbb{R}^n \setminus (H_1 \cup H_2)$, there is a quadric surface Q that contains $\{v\} \cup E_1 \cup E_2$.*

Proof. Clearly, we can choose Cartesian coordinates in \mathbb{R}^n such that

$$\begin{aligned} E &= \{(0, 0, \xi_3, \dots, \xi_n) \mid \xi_3^2 + \dots + \xi_n^2 = 1\}, \\ E_1 &= \{(\xi_1, 0, \xi_3, \dots, \xi_n) \mid (\xi_1 - \rho_1)^2 + \xi_3^2 + \dots + \xi_n^2 = \rho_1^2 + 1\}, \\ E_2 &= \{(0, \xi_2, \xi_3, \dots, \xi_n) \mid (\xi_2 - \rho_2)^2 + \xi_3^2 + \dots + \xi_n^2 = \rho_2^2 + 1\}, \end{aligned}$$

where $\rho_1 > 0$ and $\rho_2 > 0$. Then H_1 and H_2 are described by the equations $\xi_2 = 0$ and $\xi_1 = 0$, respectively. Consider the family of quadric surfaces $Q(\mu) \subset \mathbb{R}^n$ given by

$$\xi_1^2 + \cdots + \xi_n^2 + 2\mu\xi_1\xi_2 - 2\rho_1\xi_1 - 2\rho_2\xi_2 - 1 = 0,$$

where μ is a parameter. Obviously, $E_i = H_i \cap Q(\mu)$, $i = 1, 2$. The point $v = (\nu_1, \dots, \nu_n)$ belongs to $\mathbb{R}^n \setminus (H_1 \cup H_2)$ if and only if $\nu_1\nu_2 \neq 0$. Then $v \in Q(\mu_0)$ for

$$\mu_0 = (1 + 2\rho_1\nu_1 + 2\rho_2\nu_2 - \nu_1^2 - \cdots - \nu_n^2)/(2\nu_1\nu_2). \quad \square$$

Lemma 11. *If K is strictly convex and regular, then $\text{bd } K$ contains an open piece of quadric surface.*

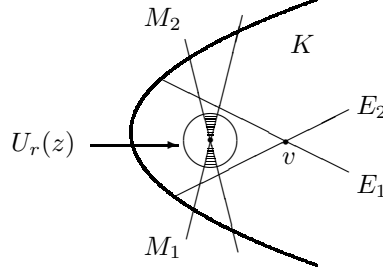
Proof. We proceed by induction on $n (\geq 3)$. Let $n = 3$. By Lemma 8, there are planes $H(u_1)$ and $H(u_2)$ such that both sections $E_1 = \text{bd } K \cap H(u_1)$ and $E_2 = \text{bd } K \cap H(u_2)$ are ellipses, with precisely two points, say v and w , in common. The set $\text{bd } K \setminus (E_1 \cup E_2)$ consists of four open pieces, at least three of them being bounded because K is line-free. We choose any of these pieces if K is bounded, and choose the piece opposite to the unbounded one if K is unbounded. Denote by Γ the chosen piece. Let L be a plane through $[v, w]$ that misses Γ and is distinct from both $H(u_1)$ and $H(u_2)$. Clearly, there is a neighborhood $\Omega \subset \text{bd } K$ of v such that for any point $z \in \Gamma \cap \Omega$, the plane L_z through z parallel to L intersects each of the ellipses E_1 and E_2 at two distinct points.

Choose a point $z \in \Gamma \cap \Omega$ and denote by P_z the plane through z that supports K (P_z is unique since K is regular), and by l_z the line through z parallel to $[v, w]$. Let \mathcal{F}_α , $\alpha > 0$, be the family of planes through l_z forming with L_z an angle of size α or less. By continuity and Lemma 7, the neighborhood Ω and the scalar α can be chosen so small that for any given plane $M \in \mathcal{F}_\alpha$, every plane $H(u)$ through the line $M \cap P_z$ intersects each of the ellipses E_1 and E_2 at two distinct points. By the same lemma, we can find a scalar $r > 0$ such that for any plane $H(u)$ through z , the convex quadric curve $\text{bd } K \cap H(u)$ intersects the closed curve $\text{bd } K \cap S_r(z)$ at two points, where $S_r(z) \subset \mathbb{R}^3$ is the sphere of radius r centered at z .

Due to Lemma 10, there is a quadric surface Q containing $\{z\} \cup E_1 \cup E_2$. By the above, given a plane $M \in \mathcal{F}_\alpha$, every plane $H(u)$ through the line $M \cap P_z$ intersects $\text{bd } K$ along an ellipse, which has five points in Q (namely, z and two on each ellipse E_i , $i = 1, 2$). Since an ellipse is uniquely defined by five points in general position, the ellipse $E(u) = \text{bd } K \cap H(u)$ lies in Q for any choice of a plane $H(u)$ through the line $M \cap P_z$, where $M \in \mathcal{F}_\alpha$. This argument shows the existence of two open “triangular” regions in $\text{bd } K \cap Q \cap U_r(z)$ which have a common vertex z and are bounded by a pair of planes $M_1, M_2 \in \mathcal{F}_\alpha$ (see the shaded sectors of $\text{bd } K \cap U_r(z)$ in the figure below). Hence the case $n = 3$ is proved.

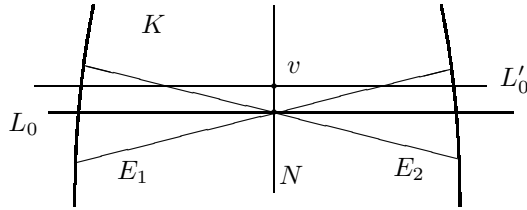
Suppose that the inductive statement holds for all $m \leq n - 1$, $n \geq 4$, and let $K \subset \mathbb{R}^n$ be a line-free, strictly convex and regular closed convex set of dimension

n that satisfies the hypothesis of Theorem 2. Since the case when K is compact is proved in [4], we may assume that K is unbounded. Then the recession cone $\text{rec } K$ contains halflines and is line-free. Choose a halfline $h \subset \text{rec } K$ with endpoint o such that the $(n-1)$ -dimensional subspace $L \subset \mathbb{R}^n$ orthogonal to h satisfies the condition $L \cap \text{rec } K = \{o\}$. Then any proper section of K by a hyperplane parallel to L is bounded (see, e.g., [12]).



Because the set $\Delta = \{\delta(u)u \mid u \in L \cap S^{n-1}\}$ is compact, we can choose a hyperplane L_0 parallel to L and properly intersecting K so far from Δ that every hyperplane $H(u)$, $u \in L \cap S^{n-1}$, intersects $\text{rint}(K \cap L_0)$. Since any section $\text{bd } K \cap H(u) \cap L_0$, $u \in L \cap S^{n-1}$, is an $(n-2)$ -dimensional convex quadric, $K \cap L_0$ satisfies the hypothesis of Theorem 2 (with L_0 instead of \mathbb{R}^n). By the inductive assumption, $\text{rbd}(K \cap L_0)$ contains a relatively open piece of an $(n-1)$ -dimensional quadric, and Lemma 9 implies that $\text{bd } K \cap L_0$ is an $(n-1)$ -dimensional ellipsoid. Let $G \subset L_0$ be an $(n-2)$ -dimensional plane through the center of $K \cap L_0$. By continuity and the argument above, there is an $\varepsilon > 0$ such that the hyperplanes L_1 and L_2 through G forming with L_0 an angle of size ε also intersect $\text{bd } K$ along $(n-1)$ -dimensional ellipsoids E_1 and E_2 , respectively. Denote by N the hyperplane through G parallel to h , and choose a point $v \in (\text{bd } K \cap N) \setminus (L_1 \cup L_2)$ so close to L_0 that the hyperplane L'_0 through v parallel to L_0 satisfies the following conditions (see the figure below):

- a) $\text{bd } K \cap L'_0$ is an $(n-1)$ -dimensional ellipsoid,
- b) L'_0 intersects the relative interiors of both $(n-1)$ -dimensional solid ellipsoids $K \cap L_1$ and $K \cap L_2$.



By Lemma 10, there is a real quadric surface Q that contains $\{v\} \cup E_1 \cup E_2$. Since the $(n-1)$ -dimensional ellipsoid $E'_0 = \text{bd } K \cap L'_0$ is uniquely determined by the set $\{v\} \cup (E_1 \cap L'_0) \cup (E_2 \cap L'_0)$, we have $E'_0 \subset \text{bd } K \cap Q$. By continuity,

there is a $\beta > 0$ such that any hyperplane L' through G that forms with L'_0 an angle of size β or less satisfies conditions a) and b) above; whence $\text{bd } K \cap L'$ is an $(n - 1)$ -dimensional ellipsoid that lies in $\text{bd } K \cap Q$. Clearly, the union of such ellipsoids $\text{bd } K \cap L'$ covers an open piece of Q that lies in $\text{bd } K$. \square

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